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STRESS SINGULARITIES AT INTERFACE CORNERS IN BONDED DISSIMILAR ISOTROPIC ELASTIC MATERIALS

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Abstract—The plane problem is considered of two materially dissimilar isotropic, homogeneous, elastic wedges that are bonded together along both of their common faces so that the cross section forms a composite full-plane with a single corner in the otherwise straight interface boundary. The loading is due to a regular plane body force field with finite resultant applied to a bounded subregion of one of the wedge domains. Emphasis is placed on determining how the order of the singularity in the stress field at the corner depends on the material constants and corner angle. Numerical results are presented for several chosen angles and all physically relevant composites. In no instance is the stress singularity more severe than that associated with the traction and displacement problems for the reentrant wedge element.

1. INTRODUCTION

THIS investigation is one of a series concerned with studying plane elastostatic solutions for cylindrical composite wedges. The emphasis in these studies has been on determining the nature of the singularity in the stress field at the apex of the constituent wedges. The study began with the solution of the traction boundary value problem for bonded materially dissimilar orthogonal wedges [1]. A discussion of [1] by Dundurs [2] showed how to reduce the number of composite material parameters involved from three to two. This reduction was significant and the problem studied in [1] was considered again in [3] with much more complete and satisfactory results. The more general geometry of bonded materially dissimilar wedges of arbitrary angles was treated in [4]. Numerical results for many particular cases of this geometry were given showing the dependence of the order of the stress singularity at the apex on the wedge angles and material combinations. In [5] the geometry of a crack terminating at a material interface was studied from the same viewpoint.

The present investigation deals with the plane problem of two dissimilar wedge regions which are bonded together along both of their common faces so that the cross section forms a composite full-plane with a single corner in the otherwise straight interface boundary (Fig. 1). The loading is due to any regular plane body force field with finite resultant applied to a bounded subregion of one of the wedge domains. The dependence of the order of the stress field singularity at the interface corner on the material constants and corner angle is displayed by use of the composite parameters α, β introduced in [2] and used also in [3–5].

Many special cases of the geometry studied here occur in engineering and material structures and hence are of interest in technology. For example, the geometry in Fig. 6 occurs as a local element at the end of a right elastic cylinder that is embedded in and bonded to a dissimilar elastic body. Such a configuration was considered recently in connection with load transfer problems by Muki and Sternberg [6]. The singular behavior

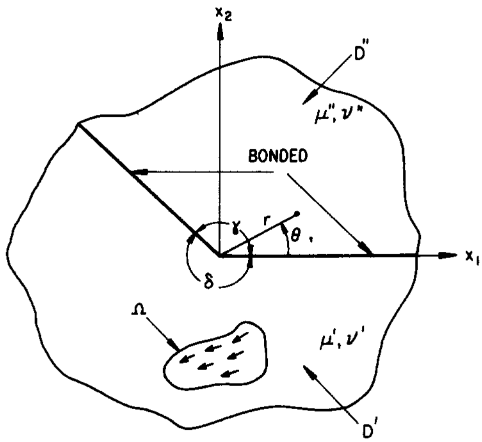

Fig. 1. Interface corner in bonded dissimilar materials.

in the plane problem indicated in Fig. 6 can be related to that in the axisymmetric case of the above three-dimensional problem. Another example where the geometric configurations studied here occur locally is provided by the microstructure of materials which are treated as homogeneous on a macroscopic scale, such as steel, but which are composed of a bonded aggregate of dissimilar angular grains on a microscopic scale. The inherent local stress concentrations in such materials may be related to the strength of the material.

2. STATEMENT OF THE PROBLEM

Let D, D'' and D' denote, respectively, the entire (x_1, x_2) plane, the wedge region $0 < \theta < \gamma$, $0 < r < \infty$ and its complementary wedge region $-\delta < \theta < 0$, $0 < r < \infty$, $(\delta = 2\pi - \gamma)$, Fig. 1. Let $\mathring{S}(\mathbf{x}; \mu, m)$ denote a two-dimensional elastostatic solution (stress and displacement fields) on D, which is occupied by an isotropic, homogeneous elastic material characterized by the two constants μ, m, corresponding to a particular body force field with finite resultant force and moment applied to a bounded subregion Ω of the region D'; μ is the shear modulus and m is related to Poisson's ratio ν by

$$m = \begin{cases} 4(1-\nu) \text{ for plane strain,} \\ \dfrac{4}{1+\nu} \text{ for generalized plane stress.} \end{cases} \tag{1}$$

We wish to obtain the two-dimensional solution corresponding to the same applied forces when D'', D' are occupied by different materials characterized by (μ'', m''), (μ', m') and are bonded together along their common interface boundary. Let $S''(\mathbf{x}; \mu', m', \mu'', m'')$, $S'(\mathbf{x}; \mu', m', \mu'', m'')$ represent this solution when $\mathbf{x}$ is in D'', D'.

It is convenient to decompose the desired solution S' on D' according to

$$S'(\mathbf{x}; \mu', m', \mu'', m'') = \mathring{S}(\mathbf{x}; \mu', m') + \bar{S}(\mathbf{x}; \mu', m', \mu'', m''). \tag{2}$$

This decomposition removes the body force loading from the residual part $\bar{S}$ of S' and, due to the assumptions with regard to its resultants, the stress field $\mathring{\tau}$ is $O(r^{-1})$ as $r \to \infty$.

In view of the decomposition (2) in which $\mathring{S}(\mathbf{x}; \mu', m')$ is supposed to be known, we need to find stress and displacement fields $\{\boldsymbol{\tau}'', \mathbf{u}''\}$, $\{\bar{\boldsymbol{\tau}}, \bar{\mathbf{u}}\}$ with plane polar components related to the Airy functions ϕ'', $\bar{\phi}$, which are suitably defined and satisfy equations of the type

$$\nabla^4 \phi = 0, \tag{3}$$

on D'', D', by equations of the form

$$\tau_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}, \qquad \tau_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2}, \qquad \tau_{r\theta} = -\frac{1}{r}\frac{\partial^2 \phi}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial \phi}{\partial \theta}, \tag{4}$$

$$\begin{aligned} \frac{\partial u_r}{\partial r} &= \frac{1}{2\mu}\left[\frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} - (1 - m/4)\nabla^2 \phi\right], \\ \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} &= \frac{1}{\mu}\left(-\frac{1}{r}\frac{\partial^2 \phi}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial \phi}{\partial \theta}\right). \end{aligned} \tag{5}$$

Because of (2), the bond conditions along the interface boundaries (continuity of traction and displacement) appear as

$$\begin{aligned} \tau''_{\theta\theta}(r, 0) &= \mathring{\tau}_{\theta\theta}(r, 0) + \bar{\tau}_{\theta\theta}(r, 0), \\ \tau''_{r\theta}(r, 0) &= \mathring{\tau}_{r\theta}(r, 0) + \bar{\tau}_{r\theta}(r, 0), \\ u''_r(r, 0) &= \mathring{u}_r(r, 0) + \bar{u}_r(r, 0), \\ u''_\theta(r, 0) &= \mathring{u}_\theta(r, 0) + \bar{u}_\theta(r, 0), \\ \tau''_{\theta\theta}(r, \gamma) &= \mathring{\tau}_{\theta\theta}(r, -\delta) + \bar{\tau}_{\theta\theta}(r, -\delta), \\ \tau''_{r\theta}(r, \gamma) &= \mathring{\tau}_{r\theta}(r, -\delta) + \bar{\tau}_{r\theta}(r, -\delta), \\ u''_r(r, \gamma) &= \mathring{u}_r(r, -\delta) + \bar{u}_r(r, -\delta), \\ u''_\theta(r, \gamma) &= \mathring{u}_\theta(r, -\delta) + \bar{u}_\theta(r, -\delta). \end{aligned} \tag{6}$$

In addition we shall require the components of the two stress fields $\boldsymbol{\tau}''$, $\bar{\boldsymbol{\tau}}$ to satisfy the regularity condition

$$\tau_{rr}, \tau_{r\theta}, \tau_{\theta\theta} = O(r^{-1+h}) \quad \text{as } r \to \infty \text{ for every } h > 0. \tag{7}$$

3. SOLUTION IN THE TRANSFORM DOMAIN

Next we apply the Mellin transform to the above field equations and bond conditions.† Let the Mellin transform of a function f, defined and suitably regular on $0 < r < \infty$, be denoted by

$$\mathscr{T}\{f; s\} = \int_0^\infty f(r) r^{s-1}\, \mathrm{d}r, \tag{8}$$

† The mathematical analysis is somewhat formal and abbreviated here since it is strictly analogous to that used in [1, 3], where several references may also be found regarding the theory of the Mellin transform and its use in the solution of elastostatic problems for wedges.

where s is a complex parameter. Let $\hat{\phi}(s,\theta)$, $\hat{t}_{ij}(s,\theta)$, $\hat{u}_i(s,\theta)$ denote the Mellin transforms with respect to r of $\phi(r,\theta)$, $r^2\tau_{ij}(r,\theta)$, $ru_i(r,\theta)$. A formal application of the Mellin transform to (3) yields an ordinary differential equation for $\hat{\phi}$ the general solution of which is

$$\hat{\phi}(s,\theta) = a(s)\sin(s\theta)+b(s)\cos(s\theta) + c(s)\sin(s\theta+2\theta)+d(s)\cos(s\theta+2\theta), \tag{9}$$

in which the functions $a(s)$, $b(s)$, etc. (a', b', etc. for $\hat{\phi}'$ and $\bar{a}$, $\bar{b}$, etc. for $\hat{\bar{\phi}}$) are to be determined through the transforms of (4) and (5) from the transform of the bond conditions (6). After use of (9) these transformed equations appear as

$$\begin{aligned} &\hat{t}_{rr}(s,\theta) = \left(\frac{d^2}{d\theta^2}-s\right)\hat{\phi}(s,\theta), \qquad \hat{t}_{\theta\theta}(s,\theta) = s(s+1)\hat{\phi}(s,\theta), \\ &\hat{t}_{r\theta}(s,\theta) = (s+1)\frac{d}{d\theta}\hat{\phi}(s,\theta), \end{aligned} \tag{10}$$

$$\begin{aligned} \hat{u}_r(s,\theta) &= \frac{1}{2\mu}[s\hat{\phi}(s,\theta)+mc(s)\sin(s\theta+2\theta)+md(s)\cos(s\theta+2\theta)], \\ \hat{u}_\theta(s,\theta) &= \frac{1}{2\mu}\left[-\frac{d}{d\theta}\hat{\phi}(s,\theta)+mc(s)\cos(s\theta+2\theta)-md(s)\sin(s\theta+2\theta)\right], \end{aligned} \tag{11}$$

$$\begin{aligned} \hat{t}''_{\theta\theta}(s,0) &= \hat{\mathring{t}}_{\theta\theta}(s,0)+\hat{\bar{t}}_{\theta\theta}(s,0), \\ \hat{t}''_{r\theta}(s,0) &= \hat{\mathring{t}}_{r\theta}(s,0)+\hat{\bar{t}}_{r\theta}(s,0), \\ \hat{u}''_r(s,0) &= \hat{\mathring{u}}_r(s,0)+\hat{\bar{u}}_r(s,0), \\ \hat{u}''_\theta(s,0) &= \hat{\mathring{u}}_\theta(s,0)+\hat{\bar{u}}_\theta(s,0), \\ \hat{t}''_{\theta\theta}(s,\gamma) &= \hat{\mathring{t}}_{\theta\theta}(s,-\delta)+\hat{\bar{t}}_{\theta\theta}(s,-\delta), \\ \hat{t}''_{r\theta}(s,\gamma) &= \hat{\mathring{t}}_{r\theta}(s,-\delta)+\hat{\bar{t}}_{r\theta}(s,-\delta), \\ \hat{u}''_r(s,\gamma) &= \hat{\mathring{u}}_r(s,-\delta)+\hat{\bar{u}}_r(s,-\delta), \\ \hat{u}''_\theta(s,\gamma) &= \hat{\mathring{u}}_\theta(s,-\delta)+\hat{\bar{u}}_\theta(s,-\delta). \end{aligned} \tag{12}$$

The substitution of (9)–(11) into (12) provides the following system of eight equations for the eight unknowns $a''(s)$, $b''(s)$, $c''(s)$, $d''(s)$, $\bar{a}(s)$, $\bar{b}(s)$, $\bar{c}(s)$, $\bar{d}(s)$;

$$\begin{aligned} &\bar{b}+\bar{d}-b''-d'' = -\hat{\mathring{t}}_{\theta\theta}(s,0)/s(s+1), \\ &s\bar{a}+(s+2)\bar{c}-sa''-(s+2)c'' = -\hat{\mathring{t}}_{r\theta}(s,0)/s+1, \\ &s\bar{b}+(s+m')\bar{d}-ksb''-k(s+m'')d'' = -2\mu'\hat{\mathring{u}}_r(s,0), \\ &-s\bar{a}-(s+2-m')\bar{c}+ksa''+k(s+2-m'')c'' = -2\mu'\hat{\mathring{u}}_\theta(s,0), \\ &-\sin(s\delta)\bar{a}+\cos(s\delta)\bar{b}-\sin(s\delta+2\delta)\bar{c}+\cos(s\delta+2\delta)\bar{d}-\sin(s\gamma)a'' \\ &\qquad -\cos(s\gamma)b''-\sin(s\gamma+2\gamma)c''-\cos(s\gamma+2\gamma)d'' = -\hat{\mathring{t}}_{\theta\theta}(s,-\delta)/s(s+1), \end{aligned} \tag{13}$$

$$
\begin{aligned}
& s\cos(s\delta)\bar{a}+s\sin(s\delta)\bar{b}+(s+2)\cos(s\delta+2\delta)\bar{c}+(s+2)\sin(s\delta+2\delta)\bar{d}-s\cos(s\gamma)a'' \\
& \quad +s\sin(s\gamma)b''-(s+2)\cos(s\gamma+2\gamma)c''+(s+2)\sin(s\gamma+2\gamma)d'' = -\hat{\tau}_{r\theta}(s,-\delta)/s+1, \\
& -s\sin(s\delta)\bar{a}+s\cos(s\delta)\bar{b}-(s+m')\sin(s\delta+2\delta)\bar{c}+(s+m')\cos(s\delta+2\delta)\bar{d} \\
& \quad -k\{s\sin(s\gamma)a''+s\cos(s\gamma)b''+(s+m'')\sin(s\gamma+2\gamma)c''+(s+m'')\cos(s\gamma+2\gamma)d''\} \\
& \quad = -2\mu'\hat{u}_r(s,-\delta), \\
& -s\cos(s\delta)\bar{a}-s\sin(s\delta)\bar{b}-(s+2-m')\cos(s\delta+2\delta)\bar{c}-(s+2-m')\sin(s\delta+2\delta)\bar{d} \\
& \quad -k\{-s\cos(s\gamma)a''+s\sin(s\gamma)b''-(s+2-m'')\cos(s\gamma+2\gamma)c''+(s+2-m'')\sin(s\gamma+2\gamma)d''\} \\
& \quad = -2\mu'\hat{u}_\theta(s,-\delta),
\end{aligned}
\tag{13 (cont.)}
$$

in which

$$k = \mu'/\mu''. \tag{14}$$

The solution of (13) together with (9)–(11) determine $\{\hat{\boldsymbol{\tau}}(s,\theta), \hat{\mathbf{u}}(s,\theta)\}$, $\{\hat{\boldsymbol{\tau}}''(s,\theta), \hat{\mathbf{u}}''(s,\theta)\}$ for $-\delta < \theta < 0, 0 < \theta < \gamma$ respectively as meromorphic functions of the complex variable s, which, in view of the regularity of $\hat{\tau}_{ij}$ and the assumptions (7), can be shown to be analytic in the open strip $-2 < \mathrm{Re}(s) < -1$ except for poles that may occur only at the zeros of the determinent of the coefficients in the system of equations (13).

The polar components of the desired solution $\{\boldsymbol{\tau}, \mathbf{u}\}$ on D' and $\{\boldsymbol{\tau}'', \mathbf{u}''\}$ on D'' are then obtained by use of the inversion theorem for the Mellin transform, which gives in the present circumstances stress and displacement components of the form

$$
\begin{aligned}
\tau_{rr}(r,\theta) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \hat{\tau}_{rr}(s,\theta) r^{-s-2}\,\mathrm{d}s, \\
u_r(r,\theta) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \hat{u}_r(s,\theta) r^{-s-1}\,\mathrm{d}s,
\end{aligned}
\tag{15}
$$

where, because of (7), the choice of c determining the path of integration in the complex line integrals is taken to be

$$c = -1-\varepsilon \qquad (0 < \varepsilon < |\mathrm{Re}(s_1)|-1), \tag{16}$$

provided s_1 denotes the location of that pole (or conjugate pair of poles if s_1 is not real) in the open strip $-2 < \mathrm{Re}(s) < -1$ with the largest real part.

4. STRESS FIELD SINGULARITIES AT THE INTERFACE CORNER

The remainder of this investigation is concerned with determining the dependence of the order of the singularity in the stress field as $r \to 0$ on the angle δ and the material constants μ'', ν'', μ', ν'.

Using arguments analogous to those in Section 4 of [1, 3] it can be shown that the stress field has a singularity of order r^{-s_1-2}, $\log r$, or 1 as $r \to 0$ according as the determinant of the coefficients in (13) has a zero s_1 in $-2 < \mathrm{Re}(s) < -1$, has no zero in this open strip but $s = -2$ is a zero of order two, has no zero in this open strip and $s = -2$ is only a simple zero. Therefore, attention is focused henceforth on the determinant of the coefficients in (13) with a view toward locating its zeros in $-2 < \mathrm{Re}(s) < -1$.

In order to express the determinant relatively concisely we make use of the composite parameters α, β introduced in [2] and used subsequently in [3–5]. Thus we recall the definitions of α, β from [3] as

$$\alpha = \frac{\mu' m'' - \mu'' m'}{\mu' m'' + \mu'' m'}, \qquad \beta = \frac{\mu'(m''-2) - \mu''(m'-2)}{\mu' m'' + \mu'' m'}. \tag{17}$$

We also make a change of complex variables

$$p = -s-1 = \xi + i\eta. \tag{18}$$

In terms of these parameters and the angle δ, the determinant of the coefficients in (13) {times the quantity $\frac{1}{4}\sin^2[p(\pi-\gamma)]\ (\alpha-\beta)^2$}, which we denote by $\mathscr{D}(\delta, \alpha, \beta; p)$, can be written as

$$\begin{aligned}\mathscr{D}(\delta, \alpha, \beta; p) = {} & [(\alpha-\beta)^2 p^2 \sin^2\delta - (1-\beta)^2 \sin^2(p\delta)][(1+\beta)^2 \sin^2(p\gamma) \\ & - (\alpha-\beta)^2 p^2 \sin^2\gamma] + (\alpha^2-1)\sin^2[p(\pi-\gamma)]\{2(\alpha-\beta)^2 p^2 \sin^2\gamma \\ & + 2(1-\beta^2)\sin(p\delta)\sin(p\gamma) - (\alpha^2-1)\sin^2[p(\pi-\gamma)]\}.\end{aligned} \tag{19}$$

The equation obtained by setting $\mathscr{D}$ equal to zero for a particular geometry, determined by the angle δ, defines a one parameter family of curves in the α–β plane with p as the parameter (or two intersecting families of curves with ξ, η as the parameters in the regions of the α–β plane where the root p is complex). Since from (19)

$$\mathscr{D}(\delta, \alpha, \beta; p) = \mathscr{D}(\gamma, -\alpha, -\beta; p), \tag{20}$$

we can confine our consideration of angles δ to $\pi \le \delta \le 2\pi$ so that $\pi \ge \gamma \ge 0$.

From (18), (19) and the asymptotic analysis of the stress field mentioned previously it follows that the order of the singularity in the stress field given in (15) and (16) as $r \to 0$ is

$$\tau'', \bar{\tau} = \begin{cases} 0(r^{-1+p_1}) \text{ if } p_1 \text{ is real,} \\ 0[r^{-1+\xi_1}\cos(\eta_1 \log r) \text{ or } r^{-1+\xi_1}\sin(\eta_1 \log r)] \\ \quad \text{if } p_1 = \xi_1 + i\eta_1 \text{ is complex,} \\ 0(\log r) \text{ if no zero of } \mathscr{D} \text{ occurs in} \\ \quad 0 < \mathrm{Re}(p) < 1 \text{ but } \mathrm{d}\mathscr{D}/\mathrm{d}p = 0 \text{ at } p = 1 \end{cases} \tag{21}$$

where p_1 is the zero of $\mathscr{D}$ in $0 < \mathrm{Re}(p) < 1$ that has the smallest real part. As will become evident $\mathscr{D}$ may have several zeros in $0 < \mathrm{Re}(p) < 1$, but p_1 determines the order of the singularity in (21) and is the only root of interest here.

5. THE ROOT p_1 OF $\mathscr{D}(\delta, \alpha, \beta; p) = 0$

The zeros of $\mathscr{D}$ are best found by plotting the loci of chosen values of the root p in the α–β plane. Thus it is desirable at this point to recall from [2, 3] some of the physical properties of the composite parameters α, β. Under the restrictions

$$0 \le \nu', \nu'' \le \tfrac{1}{2}, \qquad 0 < \mu', \mu'' < \infty, \tag{22}$$

all values of α, β determined by (1) and (17) are contained in the parallelograms in the α–β plane shown in Fig. 2. The generalized plane stress parallelogram—dashed lines—is completely contained in the one for plane strain. The four elastic constants μ', ν', μ'', ν'' with (1) and (17) determine a unique point in the α–β plane, but one point in the α–β plane

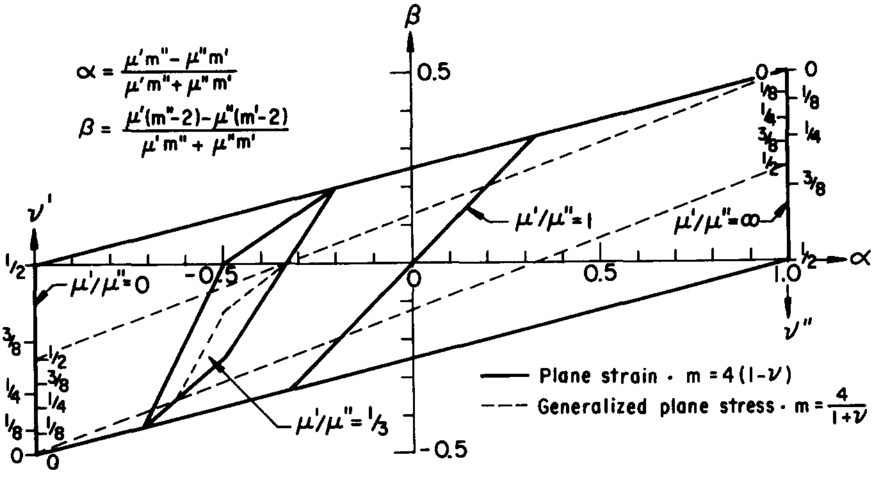

FIG. 2. Parallelograms of physically relevant material combinations.

may correspond to an infinite number of material combinations. The ratio of the two shear moduli restricts the possible values of α, β to a smaller polygon. Such a polygon is shown in Fig. 2 for $\mu'/\mu'' = \frac{1}{3}$. This polygon is degenerate to a straight line for the ratios $\mu'/\mu'' \to 0$ ($\alpha \to -1$), $\mu'/\mu'' = 1$ ($\alpha = \beta$) and $\mu'/\mu'' \to \infty$ ($\alpha \to 1$). The point $\alpha = \beta = 0$ represents identical materials. Poisson's ratio ν' increases upwardly from 0 to $\frac{1}{2}$ along the left vertical line ($\alpha = -1$) of the appropriate parallelogram and ν'' increases downwardly likewise along the right vertical ($\alpha = 1$). A choice of ν', ν'' determines a straight line across the parallelogram along which μ'/μ'' varies from 0 to ∞ and all such lines for which $\nu' = \nu''$ pass through the origin.

Zeros of $\mathscr{D}$ for certain particular composites or angles

From (19) one can verify that for $\mu'/\mu'' \to 0$, so that $\alpha \to -1$, $\mathscr{D}$ assumes the form

$$\mathscr{D}(\delta, -1, \beta; p) = [\sin^2(p\gamma) - p^2 \sin^2 \gamma]\left[p^2 \sin^2 \delta - \left(\frac{1-\beta}{1+\beta}\right)^2 \sin^2(p\delta)\right], \tag{23}$$

in which β assumes its limiting value $(2-m')/m'$ as obtained from (17) as $\mu'/\mu'' \to 0$. The first factor in (23) determines the root p appropriate to the traction boundary value problem for a single wedge of angle γ. It represents for the limit $\mu'/\mu'' \to 0$ a "free-free" wedge ($\mu' \to 0$) and coincides with equation (15) of [7]. The second factor in (23) determines the roots p appropriate to the displacement boundary value problem for a single wedge of angle δ and Poisson's ratio ν'. It represents for the limit $\mu'/\mu'' \to 0$ a "clamped-clamped" wedge ($\mu'' \to \infty$) and with the generalized plane stress interpretation of m' can be brought into agreement with equation (16) of [7]. Because of (20) the same conditions prevail for the limit $\mu'/\mu'' \to \infty$, so that $\alpha \to 1$, but with the role of the two wedges reversed.

Since the zero p_1 of $\mathscr{D}$ at $\alpha = \pm 1$ coincides with the root of the two factors in (23) with the smallest real part it is useful to have the relevant root of each of these factors for the full ranges of angles and Poisson's ratio. Figure 3(a) shows the roots, which are always real, as a function of wedge angle for the first factor in (23). The solid curve coincides with curve 1 in Fig. 1 of [7] for a free-free wedge. Figure 3(b) shows the roots, which are also always real, as a function of wedge angle and β for the second factor in (23). The solid and dashed families of lines correspond to different sets of roots. Since the solid family always corresponds to a smaller value of p it provides the relevant root. Curve 3 in Fig. 1 of [7] is supposed to give the minimum root for the second factor in (23) at $\nu' = 0{\cdot}3$, or $\beta = -0{\cdot}35$; but the values on this curve appear to coincide with those of the dashed family in Fig. 3(b) and therefore to be in error.

Since neither of the factors in (23) has a root in $0 < \mathrm{Re}(p) < 1$ for angles less than π, and since, as remarked earlier, we need consider only $\delta \geq \pi$ it follows that the root p_1 of $\mathscr{D}$ at $\alpha = -1$ is determined by Fig. 3(b) with angle δ whereas the root p_1 of $\mathscr{D}$ at $\alpha = 1$ is determined by Fig. 3(a) with angle δ. Notice that the factors in (23) are of the same form when $\beta = 0$ and that the information in Fig. 3(a) is also contained in Fig. 3(b) at $\beta = 0$.

Along the lines $\alpha = \beta$ (19) gives

$$\mathscr{D}(\delta, \beta, \beta; p) = -(1-\beta^2)^2 \sin^4(p\pi), \tag{24}$$

so that no root occurs in $0 < \mathrm{Re}(p) < 1$ but $(\mathrm{d}\mathscr{D}/\mathrm{d}p)|_{p=1}$ vanishes. Hence by (21) the line $\alpha = \beta$ (i.e. $\mu' = \mu''$) represents those composites for which the singularity in the stress field is of order $\log r$.

The particular values of p_1 at $\alpha = \pm 1$ and $\alpha = \beta$ just discussed can be observed in all the cases chosen for numerical results (see Figs. 4–9).

For the particular angle $\gamma = \delta = \pi$ (19) gives

$$\mathscr{D}(\pi, \alpha, \beta; p) = -(1-\beta^2)^2 \sin^4(p\pi), \tag{25}$$

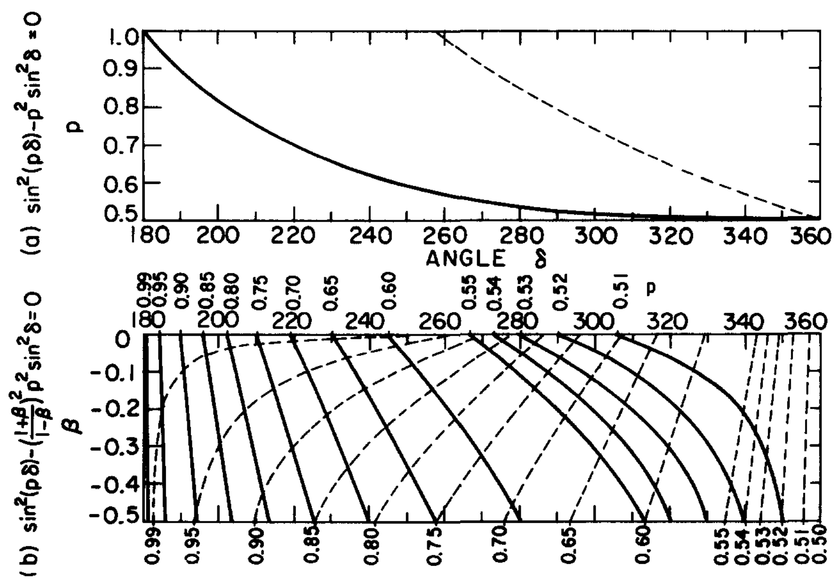

Fig. 3. Root p for (a) tractions, or (b) displacements on a single wedge.

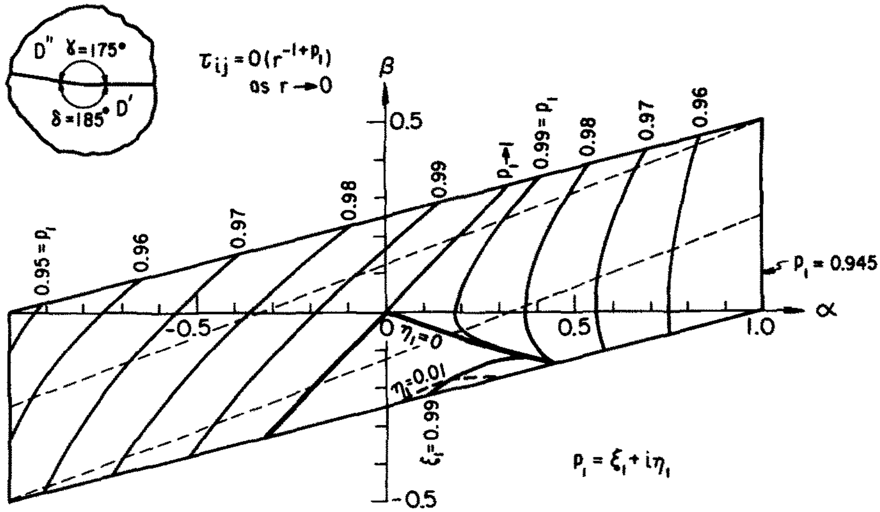

FIG. 4. Roots $p = p_1$ of $\mathscr{D}(\delta, \alpha, \beta; p) = 0$ for $\delta = 185°$.

which is the same as the form for $\alpha = \beta$ in (24) and no zero occurs in the strip $0 < \mathrm{Re}(p) < 1$ as is to be expected for bonded half-planes.

The limit $\delta \to 2\pi$ produces from (19)

$$\mathscr{D}(2\pi, \alpha, \beta; p) = -(\alpha^2-1)^2 \sin^4(p\pi), \tag{26}$$

which does not vanish for $0 < \mathrm{Re}(p) < 1$ as long as $\alpha \neq \pm 1$. However, recall from (23) and Figs. 3(a, b) that when $\alpha = \pm 1$ the root $p = \frac{1}{2}$ is associated with $\delta = 2\pi$. It is not difficult to understand the different physical interpretation that corresponds to the different results obtained here as the limits $\delta \to 2\pi$, $\alpha^2 \to 1$ are taken in different orders.

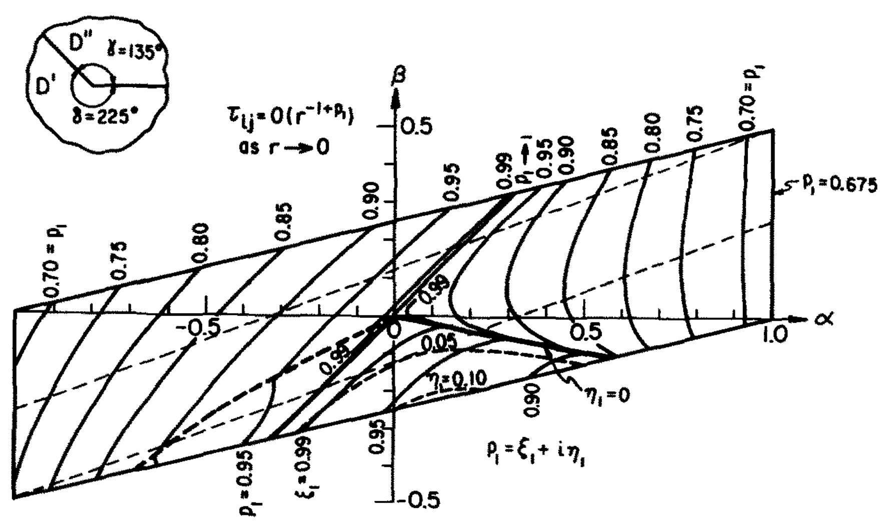

FIG. 5. Roots $p = p_1$ for $\mathscr{D}(\delta, \alpha, \beta; p) = 0$ for $\delta = 225°$.

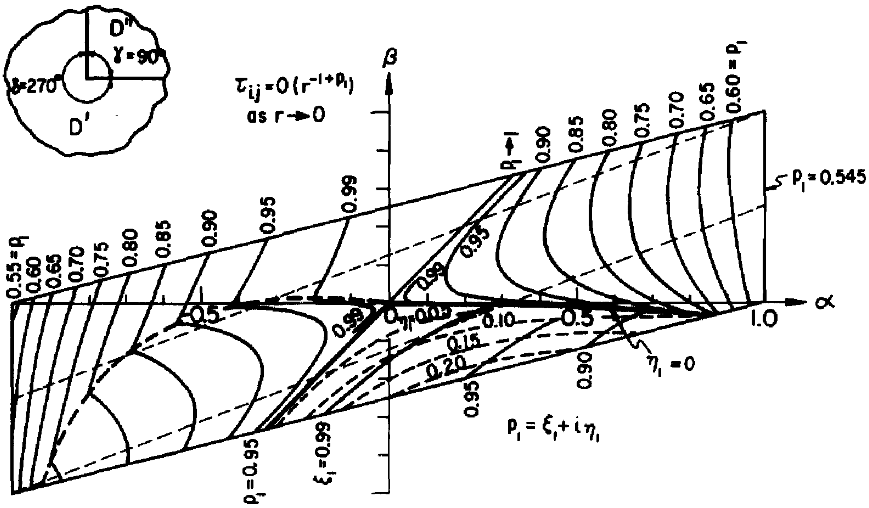

FIG. 6. Roots $p = p_1$ of $\mathscr{D}(\delta, \alpha, \beta; p) = 0$ for $\delta = 270°$.

For angles δ between π and 2π the zeros of $\mathscr{D}$ have to be found numerically from (19). For this purpose it is advantageous to divide both sides of (19) by $(\alpha-\beta)^4$ and introduce new parameters a, b through

$$a = \frac{1-\alpha}{\alpha-\beta}, \qquad b = \frac{1+\alpha}{\alpha-\beta}; \qquad \alpha = \frac{b-a}{a+b}, \qquad \beta = \frac{b-a-2}{a+b}. \tag{27}$$

Then (19) can be written in the form

$$\frac{\mathscr{D}}{(\alpha-\beta)^4} = K_1(\delta, b; p)a^2 + K_2(\delta, b; p)a + K_3(\delta, b; p), \tag{28}$$

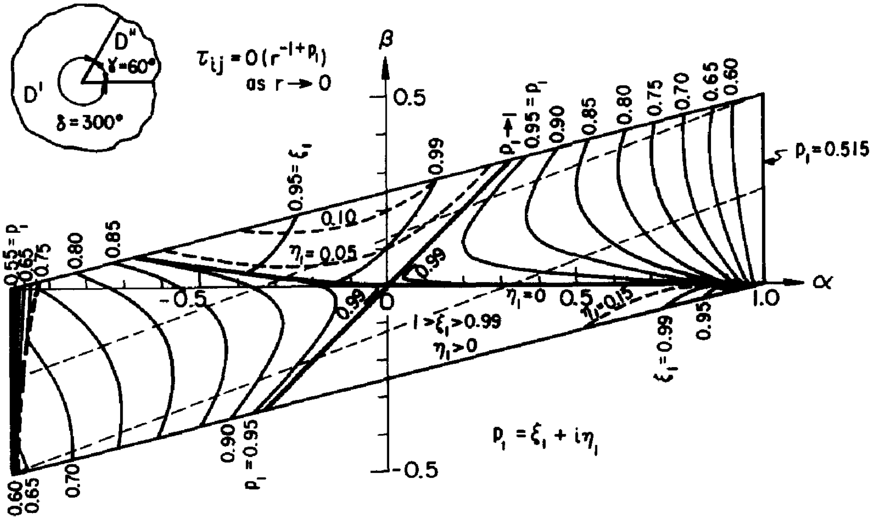

FIG. 7. Roots $p = p_1$ of $\mathscr{D}(\delta, \alpha, \beta; p) = 0$ for $\delta = 300°$.

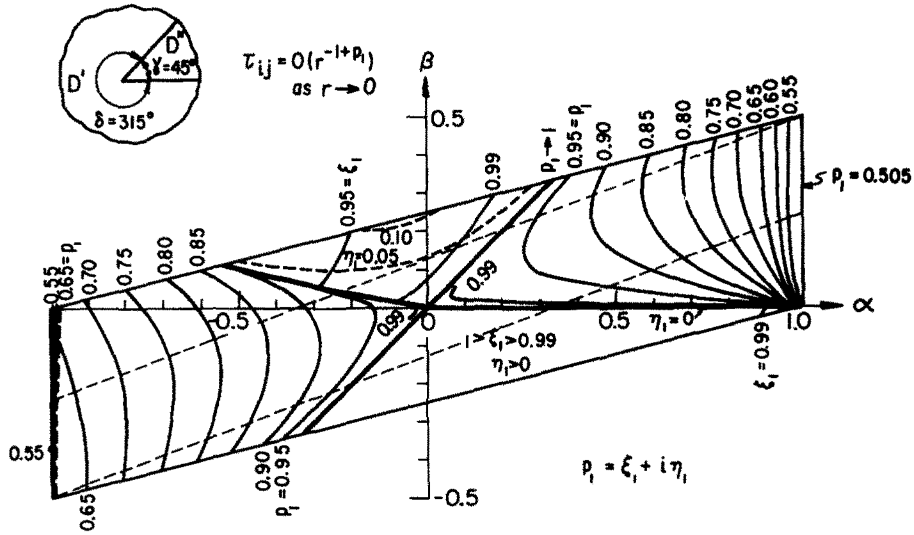

FIG. 8. Roots $p = p_1$ of $\mathscr{D}(\delta, \alpha, \beta; p) = 0$ for $\delta = 315°$.

where

$$
\begin{aligned}
K_1(\delta, b; p) &= p^2 \sin^2\gamma \sin^2(p\delta) - [b \sin^2(p\pi) - \sin(p\gamma)\sin(p\delta)]^2, \\
K_2(\delta, b; p) &= 2(1-b)\sin(p\gamma)\sin(p\delta)[b\sin^2(p\pi) - \sin(p\gamma)\sin(p\delta)] \\
&\quad + 2p^2\sin^2\gamma\{\sin^2(p\delta) - b\sin^2[p(\pi-\gamma)]\}, \\
K_3(\delta, b; p) &= [p^2\sin^2\gamma - (1-b)^2\sin^2(p\gamma)][\sin^2(p\delta) - p^2\sin^2\gamma].
\end{aligned}
\tag{29}
$$

For chosen geometry δ and real values of p it is an easy matter to solve the quadratic equation for a obtained by setting $\mathscr{D} = 0$ in (28) for given values of b. The corresponding

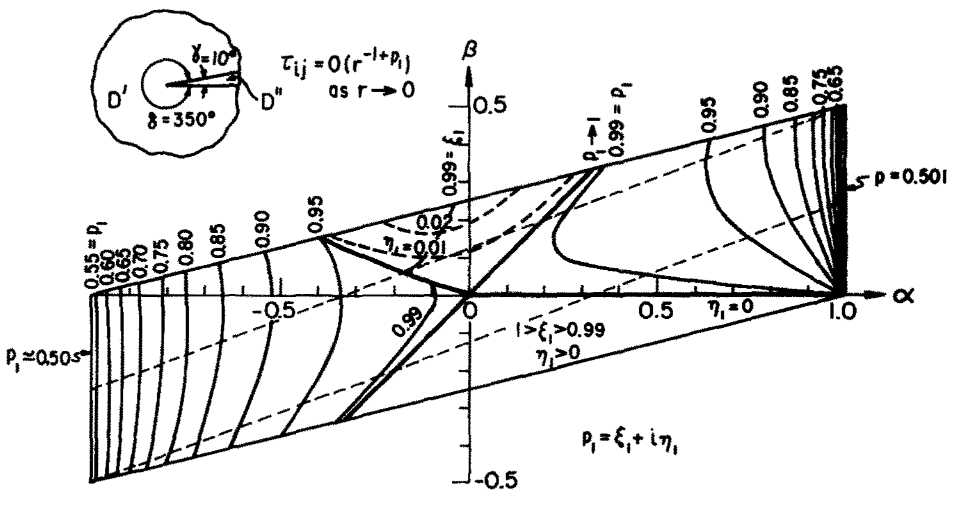

FIG. 9. Roots $p = p_1$ of $\mathscr{D}(\delta, \alpha, \beta; p) = 0$ for $\delta = 350°$.

values of α, β are then computed from (27). It usually happens that such real root loci will not pass through certain regions of the α–β plane, which means that the p is complex in those regions. When $p = \xi + i\eta$ is complex we write

$$\mathscr{D}(\delta, \alpha, \beta ; \xi + i\eta) = \mathscr{D}_R(\delta, \alpha, \beta ; \xi, \eta) + i\mathscr{D}_I(\delta, \alpha, \beta ; \xi, \eta), \tag{30}$$

and $\mathscr{D}$ vanishes for p when $\mathscr{D}_R, \mathscr{D}_I$ vanish simultaneously for ξ, η. In this manner we obtain a net of intersecting curves with ξ, η as the parameters in the complex root region of the α–β plane.

6. NUMERICAL RESULTS AND DISCUSSION

A simple computer program was written to solve for the roots of the quadratic in (28) (or the pair of quadratics when p is complex). Sufficient values of δ were chosen to reveal the dependence of p_1 on δ for $\pi \leq \delta \leq 2\pi$ and any desired composite determined by a choice of α, β. The results for $\delta = 185°$, $225°$, $270°$, $300°$, $315°$ and $350°$ are exhibited in Figs. 4–9. In each figure the root p at $\alpha = -1$ coincides with the values obtained from Fig. 3(b) with angle δ and the single value of p_1 at $\alpha = 1$ coincides with that obtained from Fig. 3(a) with angle δ. The limit $p_1 \to 1$ falls on the line $\alpha = \beta$.

The smallest values of p_1 occur always at $\alpha = \pm 1$ which means that in no instance is the stress singularity more severe than that associated with the traction and displacement problems for a single wedge of angle δ.

The general root picture does not change much in character with angle δ and it can be discussed with the aid of Fig. 6 for the case $\delta = 270°$. The region $\alpha > \beta$ is divided into a real and a complex root region. The transition line ($\eta_1 = 0$) from real to complex roots is the envelope of the real root loci for which p_1 varies from $p_1 = 1$ at $\alpha = \beta$ to $p_1 = 0{\cdot}545$ at $\alpha = 1$. The root p_1 is real in the entire $\alpha < \beta$ region. As in Fig. 3(b) there are two roots of nearly the same value for each α, β in this region. In fact, p_1 corresponds to one family over part of the region $\alpha < \beta$ and to the other family over the remainder of the region. Along the heavy dashed line the root p_1 is a double root. The situation is very much the same for $\delta = 225°$ and $\delta = 185°$ as shown in Figs. 5 and 4. The dashed double root line moves closer to the line $\alpha = \beta$ as $\delta \to \pi$. For $\delta = 185°$ the smallest value of p_1 is 0.945 at $\alpha = 1$, thus revealing the expected limit $p_1 \to 1$ for all α, β as $\delta \to \pi$.

When $\delta > 270°$ as is the case in Figs. 7–9, the double root line in the $\alpha < \beta$ region does not pass through $\alpha = \beta = 0$, and a complex root region occurs on this side of the $\alpha = \beta$ line also. We see that as $\delta \to 2\pi$ the dashed double root line approaches $\alpha = -1$ where $p_1 \to 0{\cdot}5$. Notice that for $\delta \cong 360°$, most of the α–β parallelogram is covered with $p_1 \cong 1$ curves, but as $\alpha^2 \to 1$ the value of $p_1 \to 0{\cdot}5$. The two different results for different orders of limits $\delta \to 360°$, $\alpha^2 \to 1$ discussed earlier is better understood in the light of these results.

The root p_1 at $\alpha = -1$ belongs to the same family for all values of δ, β. The intercepts of this family at $\alpha = -1$ correspond to the solid lines in Fig. 3(b). The dashed lines in this figure represent the intercepts of the other family of roots.

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Абстракт—Рассматривается плоская задача однородных, упругих клинов, изготовленных из двух разнородных, изотропных материалов. Эти клины соединенные с собой вдоль их общих сторон так, что поперечное сечение составляет составную полную плоскость с одним углом между двумя простыми поверхности раздела. Нагрузка приложенная к регулярному плоскому полю массовых сил с конечной равнодействующей силой, приложенной к ограниченному подрайону одной из областей клина. Обращается внимание на определении факта, как порядок сингулярности для поля напряжений при вершине зависит от постоянных материала и угла вершины. Даются численные результаты для некоторых выбранных углов и всех физически зависимых слоев материала. В никаком примере сингулярность напряжений более строга, чем такая, либо связанная с задачами сцепления и перемещения для входящего элемента клина.